The operators S_k, C_k and Λ_k are introduced here which are obtained from operators S, C and Λ by the substitution of operators Δ_k (3.6) for the Laplacians. The argument h is left out as in (2.2) and (2.3). The load components K_r and K_z which appear in (3.7) assume the following form after transformations (3.2)–(3.6) are applied to expressions (2.3):

$$K_r = \left\{\left[\frac{m}{2}\Lambda_2\Delta_2 - 2(m-1)S_2\right]\left(\partial_r - \frac{1}{r}\right)p_{0r} + \left[\frac{mh}{2}S_2\left(\partial_r^2 - \frac{\partial_r}{r}\right) - 2(m-2)C_2\right]p_{0z}\right\}\frac{dh}{dr} + [2(m-1)C_1 - mhS_1\Delta_1]\,p_{0r} + (m\Lambda_1\Delta_1 - 2S_1)\,\partial_r p_{0z} + 2(m-1)\,Dp_{1r}$$

$$K_z = \{[2(m-1)C_1 - mhS_1\Delta_1]\,p_{0r} + (m\Lambda_1\Delta_1 - 2S_1)\,\partial_r p_{0z}\}\frac{dh}{dr} - [(m-2)S + mhC]\left(\partial_r + \frac{1}{r}\right)p_{0r} + [2(m-1)C + mhS\Delta]\,p_{0z} - 2(m-1)\,Dp_{1z} \qquad (3.8)$$

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CONTACT PROBLEM FOR AN ELASTIC INFINITE CONE

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B. M. NULLER
(Leningrad)
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An exact solution is given herein for the mixed axisymmetric problem of elasticity theory for an infinite cone. It is assumed that the shear stresses are zero on its whole boundary surface $\theta = \theta_1$, and the homogeneous conditions for the normal stresses and normal displacements are separated by the circle $\theta = \theta_1$, $r = 1$ (r, θ, φ are spherical coordinates).

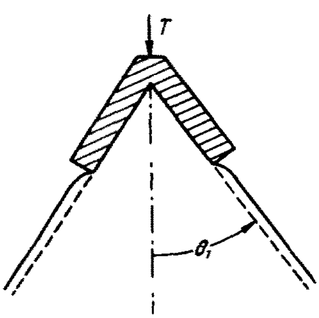

Fig. 1

Such problems arise, for example, in determining the state of stress of a cone compressed at its tip by a rigid cap of the same vertex angle as the cone (Fig. 1). They also arise in analyzing the intrusion of a conical die into a conical cavity made in an elastic space. The case $\theta_1 = {}^1/_2\,\pi$ corresponds to the symmetric indentation of a flat circular die into an elastic half-space.

It is assumed in formulating the problem that the elastic stress energy at the edge of the die and the stresses at the cone tip are bounded. As the solution shows, these conditions imply the appearance of a stress field at infinity which is statically equivalent to some axial force T.

1. We take the displacement vector components in the Gutman [1] form

$$2Gu_r = \frac{\partial \Phi}{\partial r} - 2(1-\sigma) r\Delta F, \qquad 2Gu_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} \tag{1.1}$$

where G and σ are elastic constants, and the function F satisfies the equation

$$\Delta\Delta F = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\operatorname{ctg}\theta}{r^2}\frac{\partial}{\partial \theta}\right)$$

$$\Phi = r\frac{\partial F}{\partial r} + (3-4\sigma) F$$

The conditions on the boundary surface of the cone $\theta = \theta_1$ are

$$u_\theta = \frac{1}{2Gr}\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{for } 0 \leqslant r \leqslant 1 \tag{1.2}$$

$$\sigma_\theta = \frac{1}{r}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} - \sigma\frac{\partial (r\Delta F)}{\partial r} - 2(1-\sigma)\Delta F = 0 \quad \text{for } 1 < r < \infty \tag{1.3}$$

$$\tau_{r\theta} = \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{\Phi}{r}\right) - (1-\sigma)\Delta F\right] = 0 \quad \text{for } 0 \leqslant r < \infty \tag{1.4}$$

From the formulation of the problem given above, as well as from the requirement for boundedness of the displacements it results that

$$\sigma_\theta = O[(1-r)^{\varepsilon_1 - 1}] \quad \text{for} \quad r \to 1-0, \ \theta = \theta_1 \ (\varepsilon_1 > 0) \tag{1.5}$$

$$u_\theta = O[(r-1)^{\varepsilon_2}] \quad \text{for} \ r \to 1+0, \ \theta = \theta_1 \ (\varepsilon_2 > 0) \tag{1.6}$$

$$F = O(r) \quad \text{for} \quad r \to \infty, \ F = O(r^2) \quad \text{for} \quad r \to 0 \tag{1.7}$$

Let us set $F = r^2 F_1 + F_2$, where F_1 and F_2 are harmonic functions, and let us apply the Mellin transform to (1.2)–(1.4) in the whole domain $0 < \theta < \theta_1$.

Integrating by parts and taking account of condition (1.7), we obtain

$$u(\nu, \theta) = \int_0^\infty u_\theta r^{\nu+1} dr = -(2G)^{-1} t [P_\nu'(x) A(\nu) + P_{\nu+2}'(x) B(\nu)] \tag{1.8}$$

$$\sigma(\nu, \theta) = \int_0^\infty \sigma_\theta r^{\nu+2} dr = t[(\nu+1)^2 P_\nu(x) + \operatorname{ctg}\theta P_\nu'(x)] A(\nu) +$$
$$+ [(\nu+2) t_1 P_{\nu+2}(x) + t \operatorname{ctg}\theta P_{\nu+2}'(x)] B(\nu) \tag{1.9}$$

$$\tau(\nu, \theta) = \int_0^\infty \tau_{r\theta} r^{\nu+2} dr = (\nu+2) t P_\nu'(x) A(\nu) + t_2 P_{\nu+2}'(x) B(\nu) \tag{1.10}$$

Here $x = \cos\theta$, $P_\nu(x)$ is the Legendre function of the first kind, the prime denotes the derivative with respect to θ, and ν is a complex parameter found by virtue of the condition for the existence of the integrals (1.8)–(1.10) in the strip

$$-3 < \operatorname{Re}\nu < -2, \quad t = (\nu - 2 + 4\sigma), \quad t_1 = [(\nu+1)^2 - 2(1-\sigma)]$$
$$t_2 = [(\nu+2)^2 - 2(1-\sigma)]$$

For $\theta = \theta_1$ we introduce the functions [2]

$$\sigma^+(\nu)=\int_0^1 \sigma_\theta r^{\nu+2}dr, \qquad u^-(\nu)=\int_1^\infty u_\theta r^{\nu+1}dr \tag{1.11}$$

which are, respectively, regular in the right Re $\nu > -3$ and left Re $\nu < -2$ half-planes. To determine these functions, as well as the functions $A(\nu)$ and $B(\nu)$ from the conditions of the boundary (1.2)–(1.4), we form a set of three equations ($x_1 = \cos\theta_1$)

$$(2G)^{-1}tP_\nu'(x_1)A(\nu) + (2G)^{-1}tP_{\nu+2}'(x_1)B(\nu) = u^-(\nu)$$

$$t[(\nu+1)^2P_\nu(x_1) + \mathrm{ctg}\theta_1 P_\nu'(x_1)]A(\nu) + [(\nu+2)t_1P_{\nu+2}(x_1) + t\,\mathrm{ctg}\,\theta_1 P_{\nu+2}'(x_1)] \times$$
$$\times B(\nu) = \sigma^+(\nu) \tag{1.12}$$

$$(\nu+2)tP_\nu'(x_1)A(\nu) + t_2P_{\nu+2}'(x_1)B(\nu) = 0$$

Substituting functions in the first equation which have been found from the other two

$$A(\nu) = \frac{t_2P_{\nu+2}'(x_1)\,\sigma^+(\nu)}{D_2(\nu)}, \qquad B(\nu) = -\frac{(\nu+2)\,tP_\nu'(x_1)\,\sigma^+(\nu)}{D_2(\nu)} \tag{1.13}$$

we arrive at the Wiener-Hopf equation

$$\sigma^+(\nu) = K(\nu)u^-(\nu) \tag{1.14}$$

$$K(\nu) = D_2(\nu)[D_1(\nu)]^{-1} \tag{1.15}$$

$$D_1(\nu) = -G^{-1}(1-\sigma)t(2\nu+3) \quad P_\nu'(x_1)P_{\nu+2}'(x_1)$$
$$D_2(\nu) = t[(\nu+1)^2t_2P_\nu(x_1)P_{\nu+2}'(x_1) - (\nu+2)^2t_1P_\nu'(x_1)P_{\nu+2}(x_1) +$$
$$+ 2(1-\sigma)(2\nu+3)\ \mathrm{ctg}\ \theta_1P_\nu'(x_1)P_{\nu+2}'(x_1)] \tag{1.16}$$

The function $D_1(\nu)$ has six simple zeros independent of θ_1: $\nu = 2-4\sigma,\ 0,\ -1,\ -{}^3/_2$, -2, -3, and the function $D_2(\nu)$ has two simple $\nu = 2-4\sigma, -{}^3/_2$ and two multiple $\nu = -1, -2$ zeros independent of θ_1. These zeros play an essential part by generating displacements and stresses at the cone vertex and at infinity. The function $K(\nu)$ is meromorphic, satisfies the evenness condition $K(\nu-{}^3/_2)=K(-\nu-{}^3/_2)$, and therefore, can be factored by the method of infinite products in the form

$$K(\nu)= K(-{}^3/_2)K^-(\nu)[K^+(\nu)]^{-1}, \quad K^-(\nu) = -(\nu+1)(\nu+2)\ [K^+(-\nu-3)]^{-1} \tag{1.17}$$

$$K^+(\nu) = (\nu+2)\prod_{k=1}^{\infty}\left(1+\frac{\mu}{\mu_{k1}}\right)\exp\left(\frac{-\mu}{\delta_{k1}}\right)\left[\prod_{k=1}^{\infty}\left(1+\frac{\mu}{\mu_{k2}}\right)\exp\left(\frac{-\mu}{\delta_{k2}}\right)\right]^{-1} \tag{1.18}$$

$$\mu = \nu + {}^3/_2, \qquad \mu_{k1} = \nu_{k1} + {}^3/_2, \qquad \mu_{k2} = \nu_{k2} + {}^3/_2$$

Here ν_{k1} and ν_{k2} are zeros of the functions $D_1(\nu)$ and $D_2(\nu)$ located in the right half-plane Re $\nu \geqslant -{}^3/_2$, and numbered to take account of their multiplicity; δ_{k1} and δ_{k2} are arbitrary sequences of numbers assuring the convergence of the infinite products. The structure of these products is such that the function $K^+(\nu)$ is regular, has no zeros, and is positive on the real axis in the right half-plane Re $\nu > -2-\varkappa$ ($\varkappa > 0$ and $\varkappa \to 0$ for $\theta_1 \to \pi$), the function $K^-(\nu)$ is regular, has no zeros and is negative on the real axis in the left half-plane Re $\nu < -2$. Let us evaluate the function $K(-{}^3/_2)$ and let us determine its sign.

Differentiating the identity $(\nu+1)[P_{\nu+1}(x) - xP_\nu(x)] = \sin\theta P_\nu'(x)$

with respect to ν we obtain the recursion formula

$$\left[\frac{\partial P_\nu(x)}{\partial \nu}\right]_{\nu=q} = \left\{\frac{\sin\theta}{q}\left[\frac{\partial P_\nu'(x)}{\partial\nu} - \frac{P_\nu'(x)}{q}\right] + x\frac{\partial P_\nu(x)}{\partial\nu}\right\}_{\nu=q-1} \tag{1.19}$$

from which we have for $q = 1/2$ by taking into account that $[dP_\nu(x)/d\nu]_{\nu=-1/2} = 0$

$$\left[\frac{\partial P_\nu(x)}{\partial\nu}\right]_{\nu=1/2} = -4\sin\theta P_{-1/2}(x), \quad \left[\frac{\partial P_\nu'(x)}{\partial\nu}\right]_{\nu=1/2} = -\sin\theta P_{-1/2}(x)$$

Making use of these equations and the identity $P_{-\nu-1}(x) = P_\nu(x)$, we obtain

$$K(-3/2) = \frac{GD_2^*(-3/2)}{2(1-\sigma)[P_{1/2}'(x)]^2} \tag{1.20}$$

$$D_2^*(-3/2) = \frac{\partial D_2(\nu)}{\partial\nu}\bigg|_{\nu=-3/2} = 4(\sigma-1)P_{1/2}(x_1)P_{1/2}'(x_1) +$$
$$+(\sigma - 7/8)\sin\theta_1[P_{1/2}(x_1)P_{-1/2}(x_1) - 4P_{-1/2}(x_1)P_{1/2}'(x_1)] + 4(\sigma-1)\operatorname{ctg}\theta_1[P_{1/2}'(x_1)]^2$$

Let us show that the function $K(-3/2)$ is strictly positive in the interval $0 < \theta_1 < \pi$. Evidently the functions in its numerator and denominator are continuous, and the latter is not zero by virtue of the relationship

$$P_{1/2}(x) = -3/8\sin\theta F(3/2, 5/2;\ 2;\ \sin^2\theta/2) < 0, \quad 0 < \theta < \pi$$

Let us assume $D_2^*(-3/2) = 0$ at the point $\theta_1 = \theta_1^\circ$. Then the expansion

$$D_2(\nu) = t\mu D_2^*(-3/2)\prod_{k=1}^{\infty}(1 - \mu^2/\mu_{k2}^2) \equiv 0$$

at this point contradicts, say, the identity

$$D_2(0) = 6(1-\sigma)(1-2\sigma)\ \sin\ 2\theta_1^\circ$$

Therefore, the function $K(-3/2)$ is continuous and does not vanish in $(0, \pi)$. Since it is positive at the point $\theta_1 = 1/2\pi$, where

$$K(-3/2) = G[4\pi(1-\sigma)]^{-1}\ [P_{1/2}'(0)]^{-2}$$

it is positive everywhere in $(0, \pi)$.

In order to select the sequences δ_{k1} and δ_{k2} and to estimate the growth of the function $K^+(\nu)$ at infinity, let us study the distribution of the large zeros ν_{k1} and ν_{k2}.

The zeros of the function $D_1(\nu)$ are eigenvalues of the Sturm-Liouville problem for the equations $(m = 1,2)$

$$(\sin\ \theta y')' + \lambda_m \sin\theta y = 0,\ \lambda_1 = \nu(\nu+1),\ \lambda_2 = (\nu+2)(\nu+3)$$

with the boundary condition $y' = 0$ at $\theta = \theta_1$. They are real, the asymptotic representation of the positive zeros is

$$\mu_{k1}^{(1)} = 1 + (k + 1/4)\pi\theta_1^{-1} + O(k^{-1}),\quad \mu_{k1}^{(2)} = \mu_{k1}^{(1)} - 2 \qquad (\mu_{k1}^{(m)} = \nu_{k1}^{(m)} + 3/2) \tag{1.21}$$

The large zeros of the function $D_2(\nu)$ in the right half-plane lie near the zeros of its principal part $D_3(\nu)$. We extract the function $D_3(\bar\nu)$ by using the asymptotic expansion

$$P_\nu^m(x) = \Gamma(\nu+m+1)[\Gamma(\nu+3/2)]^{-1}(1/2\ \pi\ \sin\ \theta)^{-1/2}\{\cos[(\nu+1/2)\theta$$
$$-1/4\pi + 1/2 m\pi] + 0(\nu^{-1})\} \tag{1.22}$$

Substituting (1.22) into (1.16), we obtain

$$D_2(\nu) = N(\nu)[D_3(\nu) + D_4(\nu)]$$

where $N(\nu)$ is the ratio of gamma functions which has no zeros for $\mathrm{Re}\,\nu > -{}^3/_2$

$$D_3(\nu) = \mu \sin 2\theta_1 - \cos\ (2\theta_1\mu) \tag{1.23}$$

$$D_4(\nu) = D_3(\nu)\, O(\nu^{-1}) + 2(\sigma - 1)\left\{\frac{2\mu(\nu+3)\operatorname{ctg}\theta_1}{(\nu+1)(\nu+2)^3}\left[\cos 2\theta_1 - \sin(2\theta_1\mu) + \frac{1}{2}\sin 2\theta_1 + \right.\right.$$
$$\left.\left. + \frac{1}{\nu+3}[\cos(2\theta_1\mu) + \sin 2\theta_1] - \frac{(\nu+3)}{(\nu+2)^2}[\cos(2\theta_1\mu) - \sin 2\theta_1]\right\}[1 + O(\nu^{-1})] \tag{1.24}$$

Let $(n_k - {}^3/_2)$ be a zero of the function $D_3(\nu)$, i.e.,

$$n_k \sin\ 2\theta_1 - \cos\ (2\theta_1 n_k) = 0 \tag{1.25}$$

Let us describe a circle γ_k of radius $h_k = |\,n_k\,|^{-1} \ln |\,n_k\,|$ with center at n_k.
By virtue of (1.25) we have on this circle

$$D_3(n_k - {}^3/_2 + h_k e^{i\psi}) = e^{i\psi}\,[4\theta_1^2 h_k^2 e^{i\psi} n_k \sin 2\theta_1 + 2\theta_1 h_k n_k \sin 2\theta_1 +$$
$$+ h_k \sin 2\theta_1] + n_k O(h_k^3)$$

Hence, owing to the second member in the brackets $\min |\,D_3(\gamma_k)\,| = O(\ln |\,n_k\,|)$, and $\max\,|\,D_4(\gamma_k)\,| = O(1)$ owing to the contents within the braces in (1.24).

Since $|D_3(\gamma_k)\,| > |\,D_4(\gamma_k)\,|$, the number of zeros for the functions $D_3(\nu)$ and $D_2(\nu)$ within the circle γ_k is identical by the Rouché theorem, therefore

$$\mu_{k2} = n_k + O(|\,n_k\,|^{-1} \ln |\,n_k\,|),\ \mathrm{Re}\,\mu_{k2} > 0 \tag{1.26}$$

For $\theta_1 \neq {}^1/_2\,\pi$ the number of the first real zeros of the function $D_2(\nu)$ is finite, hence, the complex zeros may always be considered large. Following [3], we set $n_k = \alpha_k + i\beta_k$ and we write (1.25) as the system

$$\alpha_k \sin 2\theta_1 = \cos(2\theta_1\alpha_k)\,\mathrm{ch}\,(2\theta_1\beta_k) \tag{1.27}$$

$$\beta_k \sin 2\theta_1 = \sin(2\theta_1\alpha_k)\,\mathrm{sh}\,(2\theta_1\beta_k) \tag{1.28}$$

For $0 < \theta_1 < {}^1/_2\pi$, $\alpha_k > 0$ and $\beta_k > 0$ we obtain $\cos(2\theta_1\alpha_k) > 0$, $\sin\ (2\theta_1\alpha_k) > 0$, from (1.27) and (1.28), which means that $2\pi_k > 2\theta_1\alpha_k < (2k+{}^1/_2)\pi$. For $\alpha_k \to \infty$ we have $\beta_k \to \infty$ from (1.27), therefore, $\sin\ (2\theta_1\alpha_k) \to 0$ from (1.28) for $\alpha_k \to \infty$ and $\alpha_k =$ $= k\pi\theta_1^{-1} + \varepsilon_k\ (\varepsilon_k > 0)$. From (1.27) we find

$$\beta_k = (2\theta_1)^{-1} \ln [2k\pi\theta_1^{-1} \sin 2\theta_1] + \varepsilon_k^*$$

The quantities ε_k and ε_k^* are on the order of $O(k^{-1} \ln k)$, taking account of the asymptotics (1.26) we obtain

$$\mu_{k2}^{(1)} = k\pi\theta_1^{-1} + i(2\theta_1)^{-1}\ \ln(2k\pi\theta_1^{-1}\ \sin\ 2\theta_1) + O(k^{-1}\ \ln k) \tag{1.29}$$

In the case of a conical cavity, i.e. for ${}^1/_2\,\pi < \theta_1 < \pi$ analogous reasoning will yield

$$\mu_{k2}^{(1)} = \frac{(2k+1)\pi}{2\theta_1} + \frac{i}{2\theta_1}\ln\left[-\frac{(2k+1)\pi}{\theta_1}\sin 2\theta_1\right] + O(k^{-1}\ln k) \tag{1.30}$$

We denote the conjugate zeros in the right half-plane by $\nu_{k2}^{(2)}$ as before $\mu_{2k}^{(1)} = \nu_{k2}^{(1)} +$ $+ {}^3/_2$, $\mu_{k2}^{(2)} = \nu_{k2}^{(2)} + {}^3/_2$. For $\theta_1 \to {}^1/_2\,\pi$ the conjugate zeros go over into multiple real zeros, and then diverge into simple zeros. For definiteness it may be considered that $\mu_{k2}^{(1)} > \mu_{k2}^{(2)}$.

Let us show that there are no other zeros for $D_2(\nu)$. Let us estimate the growth of the functions $D_3(\nu)$ and $D_4(\nu)$ along a closed contour composed of the four segments

$$|\operatorname{Im}\nu| \leqslant (2\theta_1)^{-1} \ln|4k\pi\theta_1^{-1}\sin 2\theta_1|, \qquad \operatorname{Re}\mu = \pm (k + 1/4)\pi\theta_1^{-1} \tag{1.31}$$

$$|\operatorname{Re}\mu| \leqslant (k + 1/4)\pi\theta_1^{-1}, \quad \operatorname{Im}\nu = \pm (2\theta_1)^{-1} \ln|4 k\pi\theta_1^{-1} \sin 2\theta_1| \tag{1.32}$$

On the vertical segments (1. 31)

$$|D_3(\nu)| \geqslant |\operatorname{Re} D_3(\nu)| = (k + 1/4)\pi\theta_1^{-1} \sin 2\theta_1$$

At points $\operatorname{Re}\mu = \alpha$ of the horizontal segments (1. 32), taking account of the condition $|\theta_1\alpha| < (k+1)\pi$, we obtain

$$|D_3(\nu)|^2 = \theta_1^{-2}\sin 2\theta_1[\theta_1^2\alpha^2 + 4k^2\pi^2 - 4\alpha k\pi\cos(2\theta_1\alpha)] - O(k\ln k) \geqslant \theta_1^{-2}\sin 2\theta_1$$
$$(|\theta_1\alpha| - 2k\pi)^2 - O(k\ln k) = O(k^2)$$

It is seen from (1. 24) that the modulus of the function $D_4(\nu)$ on the vertical segments is on the order of $O(1)$, and grows as $O(k/\ln k)$ on the horizontal segments. Therefore, $|D_3(\nu)| > |D_4(\nu)|$ on the whole countour chosen, and therefore, by the Rouché theorem an identical number of zeros of the functions $D_3(\nu)$, $N(\nu)$ and $D_2(\nu)$ is contained within sufficiently large rectangles. In the half plane $\operatorname{Re}\nu > -3/2$ the modulus of a zero of the function $D_3(-\nu - 3/2)$ is between the moduli of two successive zeros of the function $D_3(\nu - 3/2)$, and zeros of the function $D_2(\nu)$ are located symmetrically relative to $\nu = -3/2$. From this and from the preceding conclusion it follows that for large ν in the right half of a rectangle wherein the asymptotic formula (1. 26) is valid, the functions $D_2(\nu)$ and $D_3(\nu)N(\nu)$ have an equal number of zeros to the accuracy of one zero, and that (1. 29), (1. 30) are exhaustive.

Let us introduce two groups of zeros according to (1. 21), (1. 29) and two sequences $\delta_{k1}^{(1)} = \delta_{k1}^{(2)} = \delta_{k2}^{(1)} = \delta_{k2}^{(2)} = k\pi\theta_1^{-1}$ into each infinite product (1. 18). The general term of the series, which converges together with the second product, hence becomes

$$u_k = \frac{\mu}{\mu_{k2}^{(1)}} - \frac{\mu\theta_1}{k\pi} = \frac{\mu\,[O(\ln k) + O(k^{-1}\ln k)]}{[k\pi\theta_1^{-1} + iO(\ln k)]\,k\pi} = \mu O(k^{-2}\ln k)$$

for the first product $u_k = \mu O(k^{-2})$. Thus, both products converge absolutely, and (1. 18) becomes

$$K^+(\nu) = Q(\nu)\prod_{k=1}^{\infty}\left(1 + \frac{\mu}{\mu_{k1}^{(1)}}\right)\left(1 + \frac{\mu}{\mu_{k1}^{(2)}}\right)\left(1 + \frac{\mu}{\mu_{k2}^{(1)}}\right)^{-1}\left(1 + \frac{\mu}{\mu_{k2}^{(2)}}\right)^{-1} \tag{1.33}$$

Here the function $Q(\nu)$ takes account of a possible shift in the numbers of the zeros in the asymptotic formulas (1. 21), (1. 29) and (1. 30) with respect to their true numbers.

To investigate the growth of the function $K^+(\nu)$ let us use the method of Al'perin [3, 4]. Let us introduce three absolutely convergent products

$$M_s = \prod_{k=1}^{\infty}\left(1 + \frac{\mu}{a_1 k + b_s}\right)\exp\left(\frac{-\mu}{a_1 k}\right) = \frac{\Gamma(b_s a_1^{-1} + 1)\exp(-\gamma\mu a_1^{-1})}{\Gamma(\mu a_1^{-1} + b_s a_1^{-1} + 1)} \quad (s = 1, 2, 3) \tag{1.34}$$

in which we set

$$b_1 = 0 \quad \text{for} \quad \theta_1 < 1/2\pi, \quad b_1 = \pi(2\theta_1)^{-1} \quad \text{for} \quad \theta_1 > 1/2\pi \tag{1.35}$$
$$b_2 = -1 + \pi(4\theta_1)^{-1}, \quad b_3 = 1 + \pi(4\theta_1)^{-1}, \quad a_1 = \pi\theta_1^{-1} \quad \text{for} \quad 0 < \theta_1 < \pi$$

Let $c_{ks} = a_1 k + b_s$, and let us represent (1. 33) as

$$K^+(\nu) = Q(\nu)\prod_{k=1}^{\infty}\frac{\mu_{k2}^{(1)}\mu_{k2}^{(2)}}{c_{k1}^2}\prod_{k=1}^{\infty}\frac{c_{k2}c_{k3}}{\mu_{k1}^{(1)}\mu_{k1}^{(2)}}\prod_{k=1}^{\infty}\frac{(c_{k1}+\mu)^2}{(\mu_{k2}^{(1)}+\mu)(\mu_{k2}^{(2)}+\mu)}\prod_{k=1}^{\infty}\frac{(\mu_{k1}^{(1)}+\mu)(\mu_{k1}^{(2)}+\mu)}{(c_{k2}+\mu)(c_{k3}+\mu)}\times$$
$$\times\frac{M_2 M_3}{M_1^2} \tag{1.36}$$

The first and second infinite products converge absolutely since the kth members of the corresponding series are of the orders of $O(k^{-2}\ln^2 k)$ and $O(k^{-2})$. The third product converges absolutely and uniformly in the right half plane $\operatorname{Re}\nu > -2-\varkappa$ since by virtue of the inequalities

$$|k\pi\theta_1^{-1}+\mu| > k\pi\theta_1^{-1}-2, \quad |k\pi\theta_1^{-1}+\mu| > |\mu+2|-2$$

we have

$$|u_k| = \frac{|kO(k^{-1}\ln k)+\mu O(k^{-1}\ln k)}{|k\pi\theta_1^{-1}+\mu|^2} < \frac{O(\ln k)}{(k\pi\theta_1^{-1}-2)^2}+\frac{O(k^{-1}\ln k)}{k\pi\theta_1^{-1}-2}+O(k^{-3}\ln k) =$$
$$= O(k^{-2}\ln k)$$

The fourth product, for which $|u_k| = O(k^{-2})$ is estimated by analogous means. It is not difficult to show that by virtue of the uniform convergence of these last two products in the whole right half-plane, their limit as $\nu \to \infty$ equals unity. Consequently, the considered first part of (1.36) can be referred to the function $Q(\nu)$.

Let us estimate the second part. Utilizing the asymptotic formula

$$\frac{\Gamma(\mu+a)}{\Gamma(\mu+b)} = \mu^{a-b}\left[1+\frac{1}{2\mu}(a-b)(a+b-1)+O(\mu^{-2})\right]$$

according to (1.34) and (1.35) for $\theta_1 < {}^1/_2\,\pi$ we obtain

$$\frac{M_2M_3}{M_1^2} = \frac{\Gamma(^5/_4-\theta_1\pi^{-1})\,\Gamma(^5/_4+\theta_1\pi^{-1})\,\Gamma^2(1+\mu\theta_1\pi^{-1})}{\Gamma(^1/_4-\theta_1\pi^{-1}+\mu\theta_1\pi^{-1})\,\Gamma(\mu\theta_1\pi^{-1}+\theta_1\pi^{-1}+{}^5/_4)\,\Gamma^2(1)} =$$
$$= \mu^{-1/2}\,[O(1)+O(\mu^{-1})]$$

and this ratio has the order of $O(\mu^{1/2})$ for $\theta_1 > {}^1/_2\pi$. There remains to determine the quantity $Q(\nu)$. Let us substitute $\nu = -{}^3/_2 + i\beta$ into (1.16), and let us replace the Legendre function by the asymptotics (1.22). After some manipulation we obtain for large values of β

$$K(-{}^3/_2+i\beta) = \frac{3G\beta}{2(1-\sigma)}+O(1) \tag{1.37}$$

On the other hand, from (1.17) and (1.36) for $\theta_1 < {}^1/_2\,\pi$ we have

$$[K(-{}^3/_2)(\beta^2+{}^1/_4)]^{-1}K(-{}^3/_2+i\beta) = [K^+(-{}^3/_2+i\beta)K^+(-{}^3/_2-i\beta)]^{-1} =$$
$$= \sqrt{\beta^2+{}^9/_4}\,|Q^{-2}|+O(1) \tag{1.38}$$

Comparing (1.37) and (1.38), we find $Q(\nu)$ and obtain from (1.36)

$$K^+(\nu) = \frac{\sqrt{2(1-\sigma)K(-{}^3/_2)\nu}}{\sqrt{3G}}+O(1) \tag{1.39}$$

We again arrive at (1.39) for $\theta_1 > {}^1/_2\,\pi$ by the same means.

Let us return to the Wiener-Hopf equation (1.14)

$$\sigma^+(\nu)K^+(\nu) = K(-{}^3/_2)u^-(\nu)K^-(\nu) \tag{1.40}$$

Since its left and right sides are regular in half-planes having a common strip $-2-\varkappa < \operatorname{Re}\nu < -2$, the function $J(\nu)$ introduced by the equality

$$J(\nu) = \sigma^+(\nu)K^+(\nu) = u^-(\nu)K^-(\nu)K(-{}^3/_2) \tag{1.41}$$

is regular in the ν plane. Let us investigate its behavior at infinity. Let us utilize relationships connecting the asymptotics of the function and its Mellin transform

if $\sigma_\theta \sim A(1-r)^\eta$ for $r \to 1-0$, then

$$\sigma^+(\nu) \sim A\Gamma(\eta+1)\,\nu^{-\eta-1} \quad \text{for } \nu\to\infty,\ \operatorname{Re}\nu > -2-\varkappa \tag{1.42}$$

if $u_\theta \sim A(r-1)^\eta$ for $r \to 1+0$, then

$$u^-(\nu) \sim A\Gamma(\eta+1)\nu^{-\eta-1} \quad \text{for} \quad \nu \to \infty, \ \mathrm{Re}\,\nu < -2$$

Substituting the estimates (1.39) and (1.42) into (1.41) under the conditions (1.5), (1.6), we obtain

$$J(\nu) = O(\nu^{1/2-\varepsilon_1}) \quad \text{for} \quad \nu \to \infty, \ \mathrm{Re}\,\nu > -2-\varkappa$$

$$J(\nu) = O(\nu^{1/2-\varepsilon_2}) \quad \text{for} \quad \nu \to \infty, \ \mathrm{Re}\,\nu < -2$$

Considering $\varepsilon_1 \leqslant 1/2$, $\varepsilon_2 \leqslant 1/2$ (otherwise the solution will be zero), we obtain $J(\nu) = C$ by virtue of the generalized Liouville theorem, and therefore, from (1.41)

$$\sigma^+(\nu) = C[K^+(\nu)]^{-1} \tag{1.43}$$

According to (1.8)–(1.10), (1.13) and the Mellin inversion theorem, the displacements and stresses are expressed as

$$u_r = \frac{1}{4\pi Gi}\int_L E(\nu)\,[(\nu+1)\,t_2 P'_{\nu+2}(x_1)\,P_\nu(x) - \tag{1.44}$$

$$-(\nu+2)^2(\nu+5-4\sigma)\,P'_\nu(x_1)\,P_{\nu+2}(x)]\,r^{-\nu-2}\,d\nu$$

$$u_\theta = -\frac{1}{4\pi Gi}\int_L E(\nu)\,[t_2 P'_{\nu+2}(x_1)\,P'_\nu(x) - (\nu+2)\,tP'_\nu(x_1)\,P'_{\nu+2}(x)]\,r^{-\nu-2}\,d\nu$$

$$\sigma_\theta = \frac{1}{2\pi i}\int_L E(\nu)\,\{t_2 P'_{\nu+2}(x_1)\,[(\nu+1)^2 P_\nu(x) + \mathrm{ctg}\,\theta P'_\nu(x)] -$$

$$-(\nu+2)\,P'_\nu(x_1)\,[(\nu+2)\,t_1 P_{\nu+2}(x) + t\,\mathrm{ctg}\,\theta P'_{\nu+2}(x)]\}\,r^{-\nu-3}\,d\nu$$

$$\tau_{r\theta} = \frac{1}{2\pi i}\int_L E(\nu)\,(\nu+2)\,t_2\,[P'_{\nu+2}(x_1)\,P'_\nu(x) - P'_\nu(x_1)\,P'_{\nu+2}(x)]\,r^{-\nu-3}\,d\nu$$

$$\sigma_r = -\frac{1}{2\pi i}\int_L E(\nu)\,(\nu+2)\,\{(\nu+1)\,t_2 P'_{\nu+2}(x_1)\,P_\nu(x) -$$

$$-(\nu+2)\,[(\nu+3)(\nu+4) - 2(1+\sigma)]\,P'_\nu(x_1)\,P_{\nu+2}(x)\}\,r^{-\nu-3}\,d\nu$$

$$\sigma_\varphi + \sigma_\theta = \frac{1}{2\pi i}\int_L E(\nu)\,(\nu+2)\,\{(\nu+1)\,t_2 P'_{\nu+2}(x_1)\,P_\nu(x) -$$

$$-(\nu+2)\,[\nu(\nu+3) + 4 - 4\sigma(\nu+2)]\,P'_\nu(x_1)\,P_{\nu+2}(x)\}\,r^{-\nu-3}\,d\nu$$

$$E(\nu) = Ct[K^+(\nu)D_2(\nu)]^{-1}$$

Integrating, L passes in the positive direction along the straight line $\mathrm{Re}\,\nu = \varkappa_0 (-\varkappa - 2 < \varkappa_0 < -2)$.

We determine the constant C from the equilibrium condition. Let us examine the principal stress vector on a spherical surface $r = \rho > 1$. Let us close the contour L in (1.44) by a system of semicircles γ_k passing between the zeros of the function $D_2(\nu)$ in the right half-plane. According to the Jordan lemma, the integrals on γ_k tend to zero as k increases. Hence, by the Cauchy theorem, the stresses for $r > 1$ are expressed as the sum of residues with opposite sign, in the zeros ν_{k2}. Since the magnitude of the principal vector is independent of ρ, the stresses reaching infinity will decreases as $O(r^{-2})$, and each residue, except perhaps the residue at the point $\nu = -1$, will yield self-equilibrated homogeneous stresses. Taking into account that the number $\nu = -1$ is a simple zero of all the numerators of the integrands in (1.44), and a double zero of the function

$D_2(\nu)$ in their denominators, and utilizing the recursion formula (1.19) to resolve the corresponding indeterminacies, and the identities

$$\frac{\partial}{\partial \nu} P_\nu'(x)\,|_{\nu=-1} = \frac{\sin\theta}{1+\cos\theta}, \qquad \frac{\partial}{\partial \nu} P_\nu(x)\,|_{\nu=0} = 2\ln\cos\frac{\theta}{2}$$

we obtain the displacements and stresses at infinity

$$2Gu_r = B\left[\frac{4(1-\sigma)}{1-2\sigma}\cos\theta - 1 - \cos\theta_1\right] r$$

$$2Gu_\theta = B\sin\theta\left[\frac{1+\cos\theta_1}{1+\cos\theta} - \frac{3-4\sigma}{1-2\sigma}\right] r$$

$$\tau_{r\theta} = \frac{B\sin\theta(\cos\theta-\cos\theta_1)}{1+\cos\theta}, \qquad \sigma_r = B\left[1+\cos\theta_1 - \frac{4-2\sigma}{1-2\sigma}\cos\theta\right]$$

etc., where

$$B = \frac{C(1-2\sigma)\sin\theta_1}{K^+(-1)\,D_2^{**}(-1)(1+\cos\theta_1)\,r^2} \tag{1.45}$$

$$D_2^{**}(-1) = 2\sin\theta_1\left[(1-2\sigma) - \frac{2(1-\sigma)}{1+\cos^2\theta_1}\right] \tag{1.46}$$

The formulas written down agree to the accuracy of a factor B with the Mitchell [5] solution for an elastic cone compressed at the vertex by an axial force T. Comparing the Mitchell constant

$$B = \frac{T(1-2\sigma)}{2\pi r^2[1-\cos^3\theta_1-(1-2\sigma)\cos\theta_1(1-\cos\theta_1)]}$$

with (1.45), we find

$$C = \frac{T(1+\cos\theta_1)K^+(-1)\,D_2^{**}(-1)}{4\pi\sin\theta_1[1-\cos^3\theta_1-(1-2\sigma)\cos\theta_1(1-\cos\theta_1)]} \tag{1.47}$$

At the point $\nu = -2$ the integrands in (1.44) for the stresses have removable singularities. The integrands for the displacements at this point have a simple pole generated by a simple zero of the numerator and a double zero of the denominator. Evaluating the residues by the same means as at the point $\nu = -1$, we obtain the constant displacements at infinity

$$-\frac{u_\theta}{\sin\theta} = \frac{u_r}{\cos\theta} = \frac{2(1-\sigma)C\sin\theta_1}{G(1+\cos\theta_1)K^+(-2)\,D_2^{**}(-2)}, \quad D_2^{**}(-2) = -D_2^{**}(-1) \tag{1.48}$$

The displacements of the angular point of the cone are zero in the solution (1.44). Hence, (1.48) yields the magnitude of the axial displacement of a die (cap) subjected to the force T

$$u_0 = -\frac{2C(1-\sigma)\sin\theta_1}{G(1+\cos\theta_1)K^+(-2)\,D_2^{**}(-2)} \tag{1.49}$$

under the condition that the displacement be zero at infinity.

Let us investigate some contact zones. Let us find the normal stress distribution under the edge of the die (cap). According to boundary condition (1.3) and expressions (1.11), (1.9), (1.43), (1.39), we have

$$\sigma^+(\nu) = \sigma(\nu,\theta_1) = C[K^+(\nu)]^{-1} \sim \frac{C\sqrt{3G}}{\sqrt{2(1-\sigma)K(-^3/_2)\,\nu}} \quad \text{for} \quad \nu\to\infty$$

From this and from the assertion converse to (1.42) it follows:

$$\sigma_\theta \sim \frac{C\sqrt{3G}}{\sqrt{2\pi(1-\sigma)K(-^3/_2)(1-r)}} \quad \text{for} \ \ \theta=\theta_1,\ r\to 1-0 \tag{1.50}$$

Let us determine the shape of the free surface of an elastic body at the edge of the die (cap). From (1.42) for $\theta = \theta_1$ we obtain

$$u_\theta = \frac{1}{2\pi i}\int_L \frac{C d\nu}{K^-(\nu) K(-{}^3/_2) r^{\nu+2}} \tag{1.51}$$

Here the contour L passes to the left of the poles of the integrand. Hence, we can make the substitution $v = (\nu + 2) \ln r$ without extending it to L, and taking the straight line $Re\ v = -1$ as path of integration L_1 in the v plane independently of the quantity r. For sufficiently small $(r - 1) > 0$ the modulus of ν will be arbitrarily large on L_1 and the function $K^-(\nu)$ can be replaced by its asymptotics (1.17), (1.39). We hence obtain from (1.51)

$$u_\theta \sim \frac{C\sqrt{2(1-\sigma)(r-1)}}{2\pi\sqrt{3GK(-{}^3/_2)}}\int_L \frac{e^{-v}\,dv}{v\sqrt{v}} \tag{1.52}$$

We replace the contour L_1 according to the Cauchy theorem and the Jordan lemma by a contour L_2 consisting of a two-bank slit along the positive semi-axis and along the circle $|v| = {}^1/_2$. Utilizing the Hankel representation of the gamma function

$$(e^{2\pi i\psi} - 1)\,\Gamma(\psi) = \int_{L_2} e^{-v} v^{\psi-1}\,dv \qquad (\psi \neq 0, -1, \ldots)$$

we obtain the value of the integral in (1.52) as $4\sqrt{\pi}$, and the formula for the normal displacement

$$u_\theta \sim -C\sqrt{\frac{8(1-\sigma)(r-1)}{3\pi G\,K(-{}^3/_2)}} \qquad \text{for } \theta = \theta_1,\ r \to 1+0 \tag{1.53}$$

Let us determine the stress at the cone vertex. Let us represent the solution (1.44) in residue series taken at the zeros of the function $D_1(\nu)$ lying in the half-plane $\operatorname{Re}\nu < -2$. According to (1.15) and the equalities

$$P_\nu'(x_1) \sim -{}^1/_2\nu(\nu+1)\sin\theta_1 \ \text{ for } x_1 \to 1,\ P_{-2}'(x_1) = -\sin\theta_1,$$

$$P_{-3}'(x_1) = -{}^3/_2 \sin 2\theta_1$$

the first zero ν_1 is in the interval $(-3, -2)$ for $\theta_1 > {}^1/_2\pi$ and $\nu_1 = -3$ for $\theta_1 < {}^1/_2\pi$. In the first case the stresses at the cone vertex are infinite, and in the second case are finite

$$\sigma_\theta = Cr^{\nu-3}\partial K^+(\nu)/\partial\nu\,|_{\nu=\nu_1} \quad \text{for } \theta_1 > {}^1/_2\pi,\ r \to 0$$
$$\sigma_\theta = \sigma_r = \frac{2CG(1+\sigma)\sin\theta_1}{3(\sigma-1)K(-{}^3/_2)K^-(-3)(1-\cos\theta_1)} \quad \text{for } \theta_1 < {}^1/_2\pi,\ r \to 0 \tag{1.54}$$

Since $K^+(-2) > 0$, then $dK^+(\nu)/d\nu > 0$ for $\nu = \nu_1$. It follows from the inequalities $K^+(-1) > 0$ and $D_2^{**}(-1) < 0$ that $C < 0$ for $T > 0$. Taking these signs as well as the signs of the functions $K(-{}^3/_2) > 0$ and $K^-(-3) < 0$ into account, we obtain $\sigma_\theta < 0$ from (1.50) and (1.54), i.e. compressive stresses originate under the edge of a die and at the cone vertex. The question of the nature of the normal stresses in the remaining part of the contact surface remains open, although it seems intuitive that an elastic cone will adhere compactly everywhere to the cap, and the solution is actually realized.

2. In the case of the half-space ($\theta_1 = {}^1/_2\pi$) the function $K(\nu)$ becomes

$$K(\nu) = -\frac{G(1+\nu)(2+\nu)\Gamma(-{}^1/_2\nu)\Gamma({}^3/_2+{}^1/_2\nu)}{2(1-\sigma)\Gamma(2+{}^1/_2\nu)\Gamma({}^1/_2-{}^1/_2\nu)}$$

and it is expedient to make another, simpler factorization in place of (1.17):

$$K(\nu) = \frac{K^-(\nu)}{K^+(\nu)}, \quad K^+(\nu) = \frac{\Gamma(2+{}^1/_2\nu)}{\Gamma({}^3/_2+{}^1/_2\nu)}, \quad K^-(\nu) = -\frac{G(1+\nu)(2+\nu)\Gamma(-{}^1/_2\nu)}{2(1-\sigma)\Gamma({}^1/_2-{}^1/_2\nu)} \tag{2.1}$$

The formulas of the preceding section hence remain valid; it is only necessary to assume $K(-{}^3/_2) = 1$ therein. For example, evaluating the functions

$$D_2^{**}(-2) = 2, \quad K^+(-2) = \pi^{-1/2}, \qquad C = -T[4\sqrt{\pi}]^{-1}$$

according to (1.46), (2.1), (1.47), and substituting them into (1.49), we obtain the known formula for the indentation of a flat circular die into an elastic half-space

$$u_0 = T(1-\sigma)\ [4\pi\ G]^{-1}$$

The normal stress distribution under the die is also found easily from (1.44)

$$\sigma_\theta = -\frac{1}{2\pi i}\int_L \frac{T r^{-\nu-3}\,d\nu}{4\sqrt{\pi}K^+(\nu)} = \frac{T}{8i\sqrt{\pi}}\int_L \frac{r^{-\nu-3}\,d\nu}{\cos(^1/_2\pi\nu)\,\Gamma(2+{}^1/_2\nu)\,\Gamma(^1/_2-{}^1/_2\nu)} =$$

$$= -\frac{T}{2\pi}\sum_{k=0}^{\infty}(ir)^k P_k(0) = -\frac{T}{2\pi\sqrt{1-r^2}}$$

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TORSION OF A TRUNCATED HYPERBOLOID

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N. A. BELOVA and Ia. S. UFLIAND
(Leningrad)

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Some torsional problems are investigated which can be solved in ellipsoidal coordinates using the Mehler-Fock transformation, specially generalized for the case of an incomplete interval. The proof of the relevant inversion formula is given.

1. Formulation of the problem and its general solution. Let us consider the torsion of a two-sheeted hyperboloid of revolution, truncated at its top by an ellipsoidal surface. In degenerate ellipsoidal coordinates

$$r = c\ \operatorname{sh}\alpha\sin\beta, \qquad z = c\ \operatorname{ch}\alpha\cos\beta \tag{1.1}$$

the body which we consider occupies the region delineated by $\alpha_0 < \alpha < \infty, 0 \leqslant \beta < \beta_0$.

If the single component of an elastic displacement $v \equiv u_\varphi(\alpha,\beta)$ is taken as the basic unknown function, the problem is reduced to solving the equation [1]